Project: Preconditioned implicit DG schemes for hyperbolic systems. Application to Shallow water and Exner models.

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Multi-scale problems

Time Multi-scale problems:

- **Models**: hyperbolic systems which can modeled complex physic as nonlinear conservation laws
- **Properties propagation**: hyperbolic systems have finite propagation speed gives by the wave velocities (eigenvalues of the Jacobian).
- **Stability**: the time step is constrained by the fastest waves.
- Exemple of Multi-scale problems:
  - **Stiff problem** : $V_{max} << 1$ and $T_f = O(1)$.
  - **Multi-scale problem**: $V_{max} << V_{min}$ and $T_f = O(V_{min})$.
  - **Steady-state problem**: $V_{max} = O(1)$ and $T_f >> 1$.

Implicit scheme:

- To treat this problem, a good option: implicit scheme.
- For implicit scheme we must invert a linear system. Two solutions:
  - **exact solvers**: too greedy for fine 2D or 3D problem.
  - **iterative solvers**: the stiff or multi-scale hyperbolic systems are ill-conditioned.
- For iterative solvers, we need to robust and efficient preconditioning.

Exemple : MHD and radiative transfert

- Ideal MHD (astrophysic, nuclear fusion) :

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0 \\ \partial_t \rho \mathbf{u} + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = \mathbf{J} \times \mathbf{B} \\ \partial_t E + \nabla \cdot \left(\mathbf{u}(\rho \frac{|\mathbf{u}|^2}{2} + \frac{\gamma}{\gamma-1} p) - (\mathbf{u} \times \mathbf{B}) \times \mathbf{B} \right) = 0 \\ \partial_t \mathbf{B} + \nabla \cdot (\mathbf{u} \times \mathbf{B} - \mathbf{B} \times \mathbf{u}) = 0 \end{cases}$$

with ρ the density, $\mathbf{u}$ the velocity, $\mathbf{B}$ the magnetic field and E the energy.

time scales:

- **simulation time** T_f : 100-10000 T_a (Alfven time).
- **time step** Δt **(**$<< T_f$**)**: gives by magneto-sonic fast wave $V_f << V_a$ (V_a Alfvén speed).

- Transport equation (photon, neutron):

$$\partial_t f + c\mathbf{\Omega} \cdot \nabla f = c\sigma \left(\int_{S^2} f d\mathbf{\Omega} - f \right), \text{ approximated by } \partial_t \mathbf{U} + c\nabla \cdot F(\mathbf{U}) = -c\sigma R(\mathbf{U})$$

with $\mathbf{\Omega}$ the direction, c the light speed and σ the opacity.

time scales:

- **time step** Δt **(**$<< T_f$**)**: constrains by $\Delta t < \frac{h}{c} + \frac{1}{c\sigma}, \quad \sigma >> 1.$

Model of the project: Exner and Shallow-water equations

- Morphodynamics flows: caused by the movement of a fluid in contact with topography (example the sediment layer).
- Many environmental problems and engineering applications.
- Shallow Water + Exner equations:

$$\begin{cases} \partial_t h + \nabla \cdot (h\mathbf{u}) = 0 \\ \partial_t h\mathbf{u} + \nabla \cdot (h\mathbf{u} \otimes \mathbf{u}) + \nabla p = h\nabla b \\ \partial_t b + \zeta \nabla \cdot \mathbf{Q} = 0 \end{cases}$$

with h the height, $\mathbf{u}$ the velocity, $\mathbf{Q} = \mathbf{Q}(\mathbf{u})$ and ζ a constant which depend to the sediment coefficient porosity.

time scales:

- **time step** dt: gives by gravity waves $\lambda = \sqrt{hg}$.
- **simulation time** T_f : gives by the sedimentation behavior.

- $dt << T_f$ consequently we propose to use implicit scheme.
- The hyperbolic systems discretized with High-Order methods are ill-conditioned.

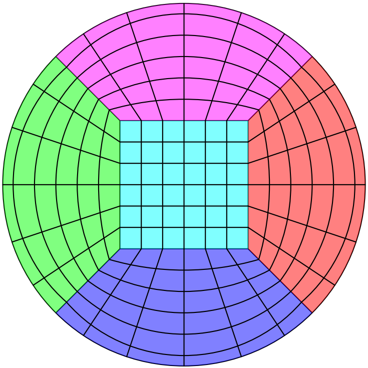
DG method for hyperbolic scheme on complex geometries

DG schemes:

- High order method adapted to discretize the hyperbolic systems.
- **Principle**: we discretize in each cell the weak form of the equations without continuity between the cells.
- **Reduction CPU**: quadrature using Gauss Lobatto points (diagonal mass matrix and quick computation of fluxes).

Complex geometries:

- **Idea**: we decompose the domain between curved macro-cells (GMSH).
- **Macro-cell**: inside the mesh is Cartesian.



Project

- Participant : E. Franck (Inria Nancy, Tonus team), P. Helluy (Inria Nancy, Tonus team IRMA) and H. Guillard (Inria Sophia, Castor team).
- Founding : IPL Fusion FRATRES

Aim :

Design efficient and robust preconditioned implicit algorithm for hyperbolic systems with DG high-order method on complex geometries

Objectives Cemracs:

- Write implicit method (based on GMRES+ Free Jacobian method) for one macro-cell (Cartesian mesh).
- Study two ways to construct physic based preconditioning.
- Validate the methods on Wave and Shallow Water equations (SH+Exner also if possible).

Post Cemracs:

- Extension to the multi macro-cells (complex geometry) case.
- Study the preconditioning (optimization, well-balanced property etc).

First preconditioning : Directional splitting

- In each macro-cell: Cartesian mesh.
- We propose to use a directional splitting to design the preconditioning.
- Exemple : advection equation

$$\partial_t u + a_x \partial_x u + a_y \partial_y u = 0$$

- Implicit scheme :

$$(I_d + \Delta t a_x \partial_x + \Delta t a_y \partial_y) u^{n+1} = u^n$$

Idea :

- Time Splitting: $(I + \Delta t a_x \partial_x + \Delta t a_y \partial_y) = (I + \Delta t a_x \partial_x)(I + \Delta t a_y \partial_y) + O(\Delta t^2)$
- Algorithm to solve Px=b (P preconditioning): $\mathbf{x}^* = (I + \Delta t a_x \partial_x)^{-1} \mathbf{b}$ and $\mathbf{x} = (I + \Delta t a_y \partial_y)^{-1} \mathbf{x}^*$
- Finit element method used for each 1D operator (jacobian more simple that with DG and less degrees of freedom).
- Exact solver for each 1D problem in the PC (tridiagonal matrices for transport and small profile matrices for hyperbolic systems).

Second preconditioning: operator splitting

Idea:

- Coupling hyperbolic problem are ill-conditioned contrary to simple diffusion and advection operators.
- **Idea**: Use operator splitting and a reformulation to approximate the Jacobian by a suitability of simple problems (advection or diffusion).
- For each subproblem we use an adapted solver as multi-grid solver.

- Implicit scheme for wave : we solve

$$\begin{cases} \partial_t u = \partial_x v \\ \partial_t v = \partial_x u \end{cases} \longrightarrow \begin{cases} u^{n+1} = u^n + \Delta t \partial_x v^{n+1} \\ v^{n+1} = v^n + \Delta t \partial_x u^{n+1} \end{cases}$$

- which is strictly equivalent to solve one parabolic problem

$$(1 - \Delta t^2 \partial_{xx}) u^{n+1} = u^n + \Delta t \partial_x v^n$$

- and applies one matrix vector product: $v^{n+1} = v^n - \Delta t \partial_x u^{n+1}$.

Preconditioning:

- The solution of $P\mathbf{x} = \mathbf{b}$ given by : $u = (1 - \Delta t^2 \partial_{xx})^{-1}(u^n + \Delta t \partial_x v^n)$ and $v = v^n - \Delta t \partial_x u^*$ with $\mathbf{x} = (u, v)$ and $\mathbf{b} = (u^n, v^n)$.
- The implicit step can be solved with multi-grid solver (small accuracy).

Second preconditioning: optimization and open questions

Models

- The operator splitting is written for the wave problem.
- **Shallow water** : we need to write the second order operator which is a reformulation of the coupling term (approximation).
- **Exner water** : how extend to the method for the Exner problem (additional equation on the topography) ?

Implementation and discretization

- Discretization of full problem: DG scheme with Gauss-Lobatto points.
- Discretization of the PC operators : EF method with the same degrees of freedom (matrices smaller and Jacobian matrices simples).
- Gmres method + Free jacobian method for the full model and multi-grid method for the PC.

Optimization

- **Adaptivity of order**: we can use a less order approximation in the preconditioning that for the full problem.

Thanks

Thanks