HLST Report: POMS project

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Outline

- POMS: *Parallel and Optimal Multigrid for B-Splines*
- Motivation and Goals.
- The Finite Element Multigrid using B-Splines.
- The Grid Transfer Operators
- The GLT Post Smoother
- First Serial Results
- Conclusions.

Motivation and Goals

- As the order of Splines increases ($p > 3$) the *convergence factor* of standard multigrid solver becomes close to 1!
- The 2 main goals of the POMS project:
  1. Implement and test the *GLT post smoother* as a solution to this problem.
  2. Parallelization of the MG+GLT solver using *distributed memory* `MPI` and *shared memory* `OpenMP` and `OpenACC`.

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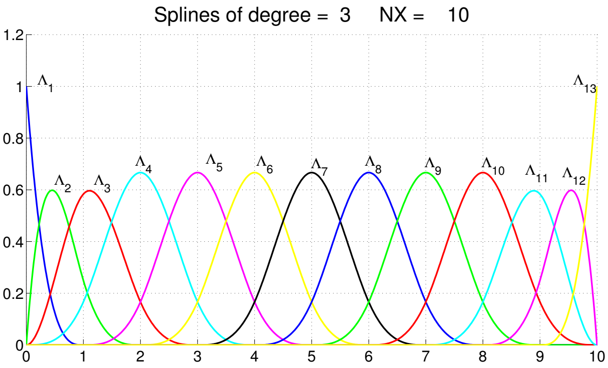


- Local support: $\mathrm{supp}(\Lambda_i^p) = [t_{i-p-1}, t_i], \quad i = 1, \ldots, N_x + p$
- $p + 1$ non zeros Splines per interval: $\Lambda_i, \Lambda_{i+1} \ldots \Lambda_{i+p}$ in $[t_{i-1}, t_i]$
- For $t_0 \leq x \leq t_{N_x}$: $\sum_j \Lambda_j^p(x) = 1$.

The standard MG V-cycle

$$\boxed{\mathbf{u}^h \leftarrow MG^h(\mathbf{u}^h, \mathbf{b}^h)}$$

1. If h is the coarsest mesh size,
   - Direct solve $\mathbf{A}^h \mathbf{u}^h = \mathbf{b}^h$
   - Goto 3.
2. Else
   - Relax $\mathbf{u}^h$ ν_1 times.
   - Restriction of the *residual* : $\mathbf{b}^{2h} \leftarrow \mathbf{R}(\mathbf{b}^h - \mathbf{A}^h \mathbf{u}^h)$.
   - $\mathbf{u}^{2h} \leftarrow MG^{2h}(\mathbf{u}^{2h}, \mathbf{b}^{2h})$ μ times.
   - Prolongation of the *error* : $\mathbf{u}^h \leftarrow \mathbf{u}^h + \mathbf{P}\mathbf{u}^{2h}$.
   - Relax $\mathbf{u}^h$ ν_2 times.
3. Return

Grid Transfer Operators

- Splines on *coarse grid*

$$\Lambda_i^{2h}(x) = \sum_{i'=1}^{N+p} R^x_{ii'} \Lambda^h_{i'}(x), \quad i = 1, \ldots, N/2+p.$$

- Restriction of **b**

$$b^{2h}_{ij} = \iint \mathrm{d}x\,\mathrm{d}y\, f(x,y)\,\Lambda_i^{2h}(x)\,\Lambda_j^{2h}(y) = \sum_{i'j'} R^x_{ii'}\,R^y_{jj'} \underbrace{\iint \mathrm{d}x\,\mathrm{d}y\, f(x,y)\,\Lambda^h_{i'}(x)\,\Lambda^{2h}_{j'}(y)}_{b^h_{i'j'}}$$

$$\Rightarrow \mathbf{b}^{2h} = \mathbf{R}^x\,\mathbf{b}^h\,(\mathbf{R}^y)^T$$

- Prolongation of **u**

$$u(x,y) = \sum_{ij} u^{2h}_{ij}\,\Lambda_i^{2h}(x)\,\Lambda_j^{2h}(y) = \sum_{i'j'} \Big[\overbrace{\sum_{ij} u^{2h}_{ij}\,R^x_{ii'}\,R^y_{jj'}}^{u^h_{i'j'}} \Big] \Lambda^h_{i'}(x)\,\Lambda^h_{j'}(y)$$

$$\Rightarrow \mathbf{u}^h = (\mathbf{R}^x)^T\,\mathbf{u}^{2h}\,\mathbf{R}^y = \mathbf{P}^x\,\mathbf{u}^{2h}\,(\mathbf{P}^y)^T$$

- The convergence of the **multigrid** depends on the **spectral features** of $\mathbf{A}^h$
- For structured matrices the spectral analysis is related to the notion of **symbol**
- **Qualitative definition:** the **symbol** is a function which describes the asymptotical spectral distribution of a matrix-sequence $\{A_n\}_n$

– GLT Symbol for the 1d stiffness/mass matrix

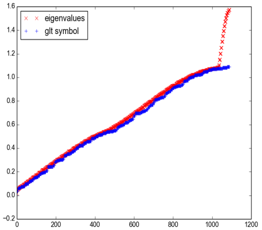
$$\{nM_n^p\}_n \sim_{\mathrm{GLT}} \mathfrak{m}_p, \quad \{\frac{1}{n}S_n^p\}_n \sim_{\mathrm{GLT}} \mathfrak{s}_p,$$

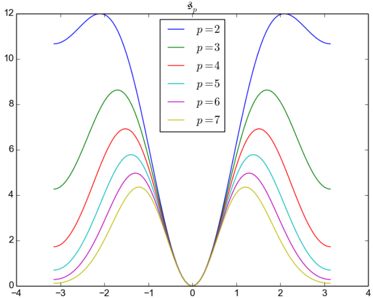
where ϕ_{2p+1} is the Cardinal Spline of degree $2p+1$ and

$$\mathfrak{m}_p(x,\theta) := \phi_{2p+1}(p+1) + 2\sum_{k=1}^{p} \phi_{2p+1}(p+1-k)\cos(k\theta).$$

$$\mathfrak{s}_p(x,\theta) := -\phi''_{2p+1}(p+1) - 2\sum_{k=1}^{p} \phi''_{2p+1}(p+1-k)\cos(k\theta).$$

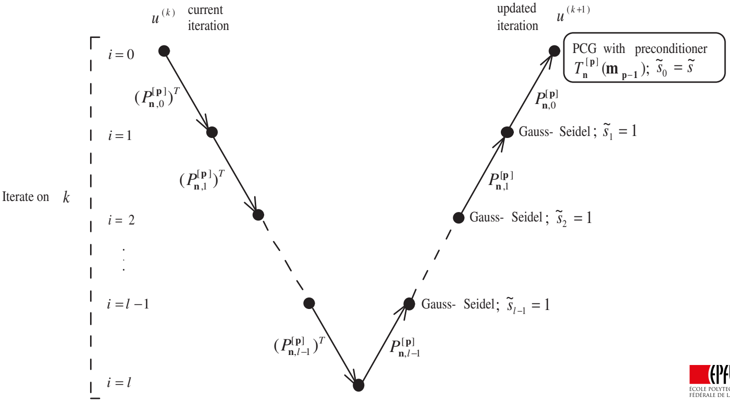
– Stiffness eigenvalues and the symbol evaluation, for a cubic spline on a 32×32 grid.





- The multigrid handles easily the exact zero at zero, but not the numerical zero at $|\theta| = \pi$, *i.e.* in high frequencies, for high spline degrees.
- we use the fact that $\mathfrak{s}_p(\theta) = (2 - 2\cos(\theta))\mathfrak{m}_{p-1}(\theta)$ and construct a preconditioner as a Banded Toeplitz matrix associated to $\mathfrak{m}_{p-1}$, *i.e.* $T(m_{p-1}(\theta))$.
- In 2d, the Post Smoother is $T(m_{p-1}(\theta_1)) \otimes T(m_{p-1}(\theta_2))$

The Multigrid V-cycle with GLT smoother



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The test problems

- The 1D problem

$$-\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}+\sigma\phi=\rho, \quad 0\le x\le L_x, \qquad \phi(0)=\phi(L_x)=0,$$

The RHS ρ is computed using the *exact solution*

$$\phi_{\mathrm{ex}}(x)=\sin\left(\frac{\pi k_1 x}{L_x}\right)+\sin\left(\frac{\pi k_2 x}{L_x}\right).$$

- The 2D problem

$$-\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]\phi=\rho \qquad 0\le x\le L_x, \quad 0\le y\le L_y,$$

$$u(0,y)=u(L_x,y)=u(x,0)=u(x,L_y)=0$$

The RHS $\rho(x,y)$ is computed using the *exact solution*

$$\phi_{\mathrm{ex}}(x,y)=\sin\left(\frac{\pi k_x x}{L_x}\right)\sin\left(\frac{\pi k_y y}{L_y}\right).$$

- Note: σ is set to 0 in the following numerical experiments.

1D Serial Results

- $N_x = 2048$, $\mathrm{tol} = 10^{-12}$
- 11 levels MG with Gauss-Seidel relaxations and $V(2,2)$ cycles.
- PCG with `SSOR` preconditionner.

ν_{psm}	$p = 10$	$p = 9$	$p = 8$	$p = 7$	$p = 6$	$p = 5$	$p = 4$	$p = 3$
0	396	174	83	41	21	11	6	5
1	94	74	29	35	16	9	5	4
2	53	30	16	12	8	6	4	4
3	40	24	14	10	8	5	4	3
4	27	19	12	8	5	4	3	3
5	20	11	7	5	4	4	3	3
6	15	9	5	4	4	4	3	3
7	13	8	5	4	4	3	3	3
8	9	5	4	3	3	3	3	3
9	6	4	3	3	3	3	3	3
10	7	4	3	3	3	3	3	3
11	6	4	3	3	3	3	3	3
12	5	3	3	3	3	3	3	3
13	4							
14	4							
15	4							
16	4							

2D Serial Results with PCG Smoother

- 256×256 grid, $\mathrm{tol} = 10^{-12}$
- 7 levels MG with Gauss-Seidel relaxation and $V(4,4)$ cycles.
- PCG with `SSOR` preconditionner.

ν_{psm}	$p=10$	$p=9$	$p=8$	$p=7$	$p=6$	$p=5$	$p=4$	$p=3$
0	>2000	1993	1160	1118	298	106	28	9
4	>2000	>2000	1302	242	51	18	7	4
8	>2000	1160	796	160	35	10	4	3
12	1892	1584	304	95	23	7	3	3
16	>2000	823	379	76	17	5	3	2
20	1346	540	288	55	12	5	3	2
24	668	863	166	42	10	4	2	
28	856	793	121	35	9	4		
32	1079	673	137	28	7	4		
36	863	468	100	24	6	3		
40	417	323	88	21	6	3		
50	585	287	66	15	5	3		
60	456	243	52	12	4	2		
70	366	178	40	10	4	2		

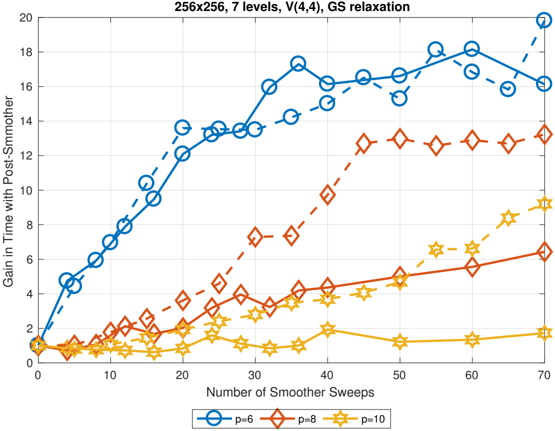
2D Serial Results with GMRES Smoother

- 256 $\times$ 256 grid, $\mathrm{tol} = 10^{-12}$
- 7 levels MG with Gauss-Seidel relaxation and $V(4,4)$ cycles.
- GMRES with `SSOR` preconditionner.

ν_{psm}	$p = 10$	$p = 9$	$p = 8$	$p = 7$	$p = 6$	$p = 5$	$p = 4$	$p = 3$
0	>2000	1993	1160	1118	298	106	28	9
5	>2000	1629	849	226	49	15	7	3
10	1310	642	411	98	26	8	4	3
15	852	298	249	63	15	6	3	2
20	575	200	155	43	10	5	2	2
25	421	134	109	30	9	4	2	2
30	329	93	62	23	8	3	2	2
35	245	75	56	18	7	3	2	2
40	213	87	39	16	6	3	2	2
45	180	58	27	13	5			
50	146	50	25	12	5			
55	98	56	24	12	4			
60	91	56	22	10	4			
65	68	43	21	9	4			
70	59	42	19	8	3			

Cost of MG with GLT Smoother

- Timing on a single `Haswell` processor.
- Solid lines for PCG smoother and dashed lines for GMRES smoother



Conclusions

- GLT post smoother scan solve the *problem of convergence* of MG using B-Splines of high order, up to $p = 10$.
- For $p > 6$, GMRES showed better performance.
- The next step is to *parallelize* the solver.